Hydraulics Overview
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July 31, 2018

• General observation & some terms
• Conservation of mass & force balance
• Flow resistance – *what sets the depth?*
• Using all the tools at once
• Energy in open channel flow  
  Rapidly varied & gradually varied flow
• Water surface, or backwater modeling
The open channel toolbox
(for flow and sediment transport)

• Conservation Relations
  Water Mass  (aka continuity)
  Momentum   (aka $F = ma$, Newton’s 2\textsuperscript{nd} Law)
  Energy

• Constitutive Relations
  Flow resistance  (we’ll use Manning eq.)
  Sediment transport  (Wednesday)

• Morphodynamics — add conservation of sediment mass
  (Wednesday/Thursday)
• Distorted view
  (channels are much wider than they are deep; $b/h = 20$ is a narrow channel)

• Width $b$, area $A$, wetted perimeter $P$
• Hydraulic radius $R = A/P$
• Mean depth $h = A/b$
• Often, $b \approx P$, so $R \approx h$
  e.g. for a rectangular channel

$$R = \frac{bh}{b + 2h} \approx \frac{bh}{b} = h$$
Continuity

- Conservation of water mass
- Just accounting, like your checkbook
- Rate of change of water mass in reach = net rate of input and output
- Inputs and outputs:
- Constant discharge: input = output, no change in water mass in reach

\[ Q_1 = U_1 A_1 \quad \text{and} \quad Q_2 = U_2 A_2 \]

For \( Q_1 = Q_2 \), \( \frac{U_2}{U_1} = \frac{A_1}{A_2} \)

*Q is water discharge (e.g. ft\(^3\)/s or m\(^3\)/s); U is mean velocity (e.g. ft/s or m/s)*
Continuity for unsteady and nonuniform 1d flow –

Now depth $h$ varies in both time and space.

For a control volume, the rate of change of mass = the net rate of mass flowing through the sides.

$$\Delta S = I - 0$$

$$\frac{\partial \rho x}{\partial t} = \rho U h B|_x - \rho U h B|_{x+\Delta x} = -\frac{\partial (\rho U h B)}{\partial x} \Delta x$$

For constant $\rho, B$. $$\frac{\partial h \Delta x}{\partial t} = -\frac{\partial (U h)}{\partial x} \Delta x$$

or $$\frac{\partial h}{\partial t} = -\frac{\partial (U h)}{\partial x}$$

$$\frac{\partial h}{\partial t} + \frac{\partial (U h)}{\partial x} = 0$$

If $\frac{\partial}{\partial t} = 0$, we recover $U h B = Q = \text{constant}$.
Momentum: Newton’s 2\textsuperscript{nd} Law, $\Sigma F = ma$

Normal flow
no accelerations, so $\Sigma F = 0$

Volume of water: $AL$
Weight of water: $\rho gAL$
Downslope component of weight of water: $\rho gALS$

$\tau_o$ is the boundary shear stress - the flow force per unit area - it drives the sediment transport

Boundary stress: $\tau_o$
(stresss is force/area)

Boundary force: $\tau_o PL$

$\tau_o PL = \rho gALS$
$\tau_o = \rho gRS$

The ‘depth-slope’ product

Figure adapted from WINXSPro manual, USFS
1b. Boundary shear stress — Unsteady, Nonuniform Flow

We still use Newton’s second law, \( \Sigma F = ma \)

But the flow is accelerating in (1) space and (2) time. These accelerations are balanced by the sum of forces, to which we now must add a third force, due to the difference in flow depth down the reach

\[
g \sin(\alpha) \approx -gS
\]

\[
P \approx \rho gh / 2
\]
If the flow is predominantly one-dimensional (no rapid changes in width or depth that give rise to vertical or lateral accelerations), we get the 1d St. Venant Eqn:

\[ ma = F \]

\[ \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} = gS - g \frac{\partial h}{\partial x} - \frac{\tau_o}{\rho R} \]

An approximate derivation is in the primer

"local" acceleration
"convective" acceleration
body force
pressure force
boundary force

Rearrange to solve for \( \tau_o \)

\[ \frac{\tau_o}{\rho g R} = S_o - \frac{\partial y}{\partial x} - \frac{V}{g} \frac{\partial V}{\partial x} - \frac{1}{g} \frac{\partial V}{\partial t} \]

Steady, uniform flow
Steady, nonuniform flow
Unsteady, nonuniform flow
We can rearrange the St. Venant eqn, to solve for shear stress

\[ \tau_0 = \rho g R \left( S - \frac{\partial h}{\partial x} - \frac{U}{g} \frac{\partial U}{\partial x} - \frac{1}{g} \frac{\partial U}{\partial t} \right) \]

1. For steady, uniform flow, we recover the depth-slope product
2. But flow is never perfectly steady and nonuniform!
3. So, how do we know whether the other terms are important or not?
4. Are we making an assumption or an approximation?

NOTE:

\[ \tau_0 = \rho g R \left( S - \frac{\partial h}{\partial x} - \frac{U}{g} \frac{\partial U}{\partial x} \right) = \rho g R S_f \]

Backwater programs (HEC-RAS) compute:

\[ S_f = \frac{d}{dx} \left( z_b + h + \frac{U^2}{2g} \right) \]

\( \rightarrow \text{flood.xls} \)
Flow from a tributary at Pt a increases rapidly from $Q_1$ to $Q_2$ over $\Delta t$. At time $\Delta t$, any increase in flow has not yet reached a distance $\Delta x$ downstream. The flow is unsteady (it is increasing with time) and it is nonuniform (it is increasing in the upstream direction).

Question: do the unsteady and nonuniform terms contribute significantly to $\tau_o$?

Given a rectangular channel:

<table>
<thead>
<tr>
<th>Channel width $b$</th>
<th>Channel slope $S_0$</th>
<th>$Q_1$</th>
<th>$Q_2$</th>
<th>$\Delta t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15 (m)</td>
<td>0.001</td>
<td>10 (m³/s)</td>
<td>15 (m³/s)</td>
<td>3600 (s)</td>
</tr>
</tbody>
</table>

Flow is unsteady and nonuniform, so we ask which terms in the St Venant Eqn are significant.

How does the shear stress change, approximately, over this reach of nonuniform flow?

We need to determine depth and velocity at the upstream and downstream end of the reach.

<table>
<thead>
<tr>
<th>Upstream</th>
<th>Downstream</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_u$</td>
<td>$0.87$ (m)</td>
</tr>
<tr>
<td>$R_u$</td>
<td>$0.98$ (m/s)</td>
</tr>
<tr>
<td>$U_u$</td>
<td>$0.89$ (m/s)</td>
</tr>
</tbody>
</table>

We also need the mean velocity and hydraulic radius for the reach:

| $R$ | $0.88$ (m) |
| $U$ | $0.83$ (m/s) |
| $\Delta h$ | $-0.253$ (m) |
| $\Delta U$ | $-0.12$ (m/s) |

Now we can compute all of the components in the brackets of the St Venant eqn

$$
\tau_0 = \rho g R \left[ S_0 - \frac{dh}{dx} - \frac{U}{g} \frac{dU}{dx} - \frac{1}{g} \frac{dU}{dt} \right]
$$

To make approximation, express in difference form

$$
\tau_0 = \rho g R \left[ S_0 - \frac{\Delta h}{\Delta x} - \frac{U}{g} \frac{\Delta U}{\Delta x} - \frac{1}{g} \frac{\Delta U}{\Delta t} \right]
$$

Note that a number of cells are named and that those names are used in formulas.

Enter Data only in green cells

Flow is unsteady and nonuniform, so we ask which terms in the St Venant Eqn are significant.

How does the shear stress change, approximately, over this reach of nonuniform flow?

We need to estimate the magnitude of the different terms in the St Venant Eqn.

The length of the flood wave $\Delta x$ is found from $U_f = \Delta x/\Delta t$, where $U_f$ is the speed of the flood wave and is approximately 1.5 times mean velocity in the reach $U = \sqrt{S_0/R}$ and $Q = UA$

so, for a rectangular channel ($A = bh$, $P = b + 2h$)

$$
Q = \sqrt{S_0} \frac{(bh)^{2/3}}{n} \quad (b + 2h)^{2/5}
$$

which can be solved iteratively for $h$

Note that a number of cells are named and that those names are used in formulas.

To make approximation, express in difference form

$$
\tau_0 = \rho g R \left[ S_0 - \frac{\Delta h}{\Delta x} - \frac{U}{g} \frac{\Delta U}{\Delta x} - \frac{1}{g} \frac{\Delta U}{\Delta t} \right]
$$

To make approximation, express in difference form

$$
\tau_0 = \rho g R \left[ S_0 - \frac{\Delta h}{\Delta x} - \frac{U}{g} \frac{\Delta U}{\Delta x} - \frac{1}{g} \frac{\Delta U}{\Delta t} \right]
$$
Use St. Venant to consider sediment transport for a steady river flow entering a reservoir. Assume a wide channel, so $R = h$. As flow enters the reservoir, it deepens and slows (by continuity!). Which effect wins out?

$$
\tau_0 = \rho g R \left( S - \frac{\partial h}{\partial x} - \frac{V}{g} \frac{\partial V}{\partial x} - \frac{1}{g} \frac{\partial V}{\partial t} \right)
$$

Consider $\left( -\frac{\partial h}{\partial x} - \frac{U}{g} \frac{\partial U}{\partial x} \right)$ for the case of constant width $b$ (for simplicity)

such that $q = U h = \text{const}$ and $\frac{\partial q}{\partial x} = 0 = \frac{\partial (U h)}{\partial x} = U \frac{\partial h}{\partial x} + h \frac{\partial U}{\partial x}$ such that

$$
\frac{\partial U}{\partial x} = -\frac{U}{h} \frac{\partial h}{\partial x}.
$$

Replace $\frac{\partial U}{\partial x}$ in the first equation above, giving

$$
\left( -\frac{\partial h}{\partial x} + \frac{U}{g} \frac{\partial h}{\partial x} \right) = \frac{\partial h}{\partial x} \left( \frac{U^2}{gh} - 1 \right).
$$

Because $\frac{U^2}{gh} < 1$ in a reservoir,

the entire quantity $\left( -\frac{\partial h}{\partial x} - \frac{U}{g} \frac{\partial U}{\partial x} \right)$ is negative. Because $\frac{\partial h}{\partial x}$ is itself positive

for flow entering a reservoir, we see that $\tau_o$ is reduced from $\rho g h S$.

A smaller value of $\tau_o$ means that transport capacity decreases and sediment deposits.
Consider a typical problem:
Specify a channel, with known geometry
(fluid $\rho$ and acceleration of gravity $g$ are also known).
For a specified discharge $Q$, what is the flow depth $h$ for steady uniform flow?
Continuity gives us $U$ (when $A(h)$ is known).
Momentum gives $\tau_o$ (when $R(h)$ is known).
Energy gives $S_f = S_o$ for steady uniform flow.
We need one more relation to find depth $h$!
This is a flow resistance relation (a constitutive relation).

![Diagram of flow in a channel]

- $S$: slope
- $B$: top width
- $h$: depth
- $P$: wetted perimeter
- $A$: x/s area

**Mean Depth** = \[ \frac{\text{Area}}{\text{Top Width}} \]

**Hydraulic Radius** = \[ \frac{\text{Area}}{\text{Wetted Perimeter}} \]
Flow Resistance

- A relation between velocity, flow depth, boundary stress, and boundary roughness
- We’ll use Manning’s eqn.
- Prefer an eqn. with constant roughness (n)
- Using continuity:

\[ U = \frac{\sqrt{S}}{R^{2/3}} \]

\[ Q = UA = \frac{\sqrt{S}}{n} AR^{2/3} \]

*S is the slope of the ‘energy grade line’, which is also the slope of the channel for uniform flow*
• For a useful approximation, consider a wide \((R \approx h)\) rectangular channel

• Solve for depth

• In field, relation between \(Q, A, \text{ & } R\) more complex, though not more complicated

• In practice, a flow resistance relation is often used to find the flow depth for a specified flow in a given channel

\[ Q = \frac{\sqrt{S}}{n} AR^{2/3} \]

\[ Q = \frac{\sqrt{S}}{n} (bh)h^{2/3} \]

\[ Q = \frac{\sqrt{S}}{n} bh^{5/3} \]

\[ n = \frac{b\sqrt{Sh^{5/3}}}{Q} \]

\[ Q = \frac{b\sqrt{Sh^{5/3}}}{n} \]

\[ h = \left(\frac{nQ}{b\sqrt{S}}\right)^{3/5} \]

\textit{Manning’s Eq., 3 ways:}
The hard part: determining roughness ‘n’

• How to measure:
  survey cross section and reach, which gives S and A, R as a function of water surface elevation (aka stage or WSEL)

• Measure discharge at a known stage, solve for $n$

\[ Q = \frac{\sqrt{S}}{n} A R^{2/3} \]

\[ n = \frac{\sqrt{S}}{Q} R^{2/3} A \]

• If you don’t measure, you are guessing
• Picture books, tables
• Formulas $n = fn(D, R, \text{veg.}, \ldots)$
Roughness Characteristics of Natural Channels

By HARRY H. BARNES, Jr.

U.S. GEOLOGICAL SURVEY WATER-SUPPLY PAPER 1849

Color photographs and descriptive data for 50 stream channels for which roughness coefficients have been determined

USGS
science for a changing world

n = 0.045; 0.073

10–1550. Provo River near Hailstone, Utah

Gage location.—Lat 40°36', long 111° 22', in SE¼ sec. 34, T. 2 S., R. 5 E., on right bank 3 miles upstream from Ross Creek and Hailstone. Section 1 is about 120 ft upstream from gage.

Drainage area.—233 sq mi.

Date of flood.—June 13, Oct. 7, 1952.

Gage height.—4.66 ft, 1.58 at gage; 5.66 ft, 2.14 ft at section 1.

Peak discharge.—1,200 cfs, 64.8 cfs.

Computed roughness coefficient.—Manning \( n = 0.045; 0.073 \).

Description of channel.—Bed and banks consist of smooth rounded rocks as much as 1 ft in diameter. Some undergrowth is below water elevations of June 13.

Reach properties

<table>
<thead>
<tr>
<th>Section</th>
<th>Area (sq ft)</th>
<th>Top width (ft)</th>
<th>Mean depth (ft)</th>
<th>Hydraulic radius (ft)</th>
<th>Mean velocity (ft per sec)</th>
<th>Length (ft) between sections</th>
<th>Fall (ft) between sections</th>
</tr>
</thead>
<tbody>
<tr>
<td>June 13, 1952</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>184</td>
<td>47</td>
<td>3.9</td>
<td>3.70</td>
<td>6.52</td>
<td>88</td>
<td>0.67</td>
</tr>
<tr>
<td>3</td>
<td>171</td>
<td>49</td>
<td>3.5</td>
<td>3.33</td>
<td>7.02</td>
<td>88</td>
<td>1.04</td>
</tr>
<tr>
<td>5</td>
<td>173</td>
<td>55</td>
<td>3.1</td>
<td>3.02</td>
<td>6.95</td>
<td>109</td>
<td>1.04</td>
</tr>
<tr>
<td>7</td>
<td>173</td>
<td>48</td>
<td>3.6</td>
<td>3.43</td>
<td>6.95</td>
<td>117</td>
<td>1.10</td>
</tr>
<tr>
<td>9</td>
<td>183</td>
<td>55</td>
<td>3.3</td>
<td>3.22</td>
<td>6.56</td>
<td>116</td>
<td>1.04</td>
</tr>
<tr>
<td>Oct. 7, 1952</td>
<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>36</td>
<td>38</td>
<td>1.0</td>
<td>0.95</td>
<td>1.79</td>
<td>88</td>
<td>0.32</td>
</tr>
<tr>
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<td>38</td>
<td>34</td>
<td>1.1</td>
<td>1.10</td>
<td>1.70</td>
<td>88</td>
<td>.84</td>
</tr>
<tr>
<td>5</td>
<td>34</td>
<td>32</td>
<td>1.1</td>
<td>.82</td>
<td>1.90</td>
<td>109</td>
<td>.84</td>
</tr>
<tr>
<td>7</td>
<td>34</td>
<td>39</td>
<td>.9</td>
<td>.86</td>
<td>1.91</td>
<td>117</td>
<td>1.28</td>
</tr>
<tr>
<td>9</td>
<td>31</td>
<td>41</td>
<td>.8</td>
<td>.76</td>
<td>2.08</td>
<td>116</td>
<td>1.12</td>
</tr>
</tbody>
</table>

Notes.—
No. 769 downstream from section 3, Provo River near Hailstone, Utah.

No. 770 upstream from section 3, Provo River near Hailstone, Utah.
Channel $n$ Values

The most important factors that affect the selection of channel $n$ values are:

1. the type and size of the materials that compose the bed and banks of the channel
2. the shape of the channel.

Cowan (1956) developed a procedure for estimating the effects of these factors to determine the value of $n$ for a channel. The value of $n$ may be computed by

$$n = (n_b + n_1 + n_2 + n_3 + n_4)m \quad (3)$$

where:

- $n_b$ = a base value of $n$ for a straight, uniform, smooth channel in natural materials
- $n_1$ = a correction factor for the effect of surface irregularities
- $n_2$ = a value for variations in shape and size of the channel cross section,
- $n_3$ = a value for obstructions
- $n_4$ = a value for vegetation and flow conditions
- $m$ = a correction factor for meandering of the channel
Combining the tools

Relating $Q$ to $\tau$ in a simple, prismatic, wide channel

This is how we link transport rates (a function of $\tau$) to water discharge $Q$

\[
Q = BhU
\]

\[
B = \alpha Q^\beta
\]

\[
h = \tau_o / \rho g S
\]

\[
U = \frac{\sqrt{S}}{n} h^{2/3} \quad \text{so}
\]

\[
Q = \left(\alpha Q^\beta\right) \left(\frac{\tau}{\rho g S}\right)^{5/3} \frac{\sqrt{S}}{n}
\]

or

\[
Q = \left[\frac{\alpha \sqrt{S}}{n} \left(\frac{\tau}{\rho g S}\right)^{5/3}\right]^{1/1-\beta}
\]

Discharge = width * depth * velocity  

**continuity**

Width increases with discharge  

**channel geometry**

The ‘depth-slope’ product  

**momentum**

Manning’s eqn  

**flow resistance**

Combine. It’s only algebra!

Boundary stress is related to discharge, slope, roughness, channel shape

This solves the **FLOW** problem
Why?

Because we generally relate changes in the bed and banks – *erosion and deposition* – to the bed shear stress \( \tau_o \) and we will want to be able to relate \( \tau_o \) to the water discharge e.g. at what \( Q \) will \( \tau_o \) be sufficient to move river bed material?

For example, if the grain size of the river bed is used to find the critical shear stress for incipient motion \( \tau_c \), this relation gives us the critical discharge \( Q_c \) needed to move the river bed.

\[
Q = BhU \\
B = \alpha Q^\beta \\
h = \frac{\tau}{\rho g S} \\
U = \frac{\sqrt{S}}{n} h^{2/3} \quad \text{so} \\
Q = \left( \alpha Q^\beta \right) \left( \frac{\tau}{\rho g S} \right)^{5/3} \frac{\sqrt{S}}{n} \quad \text{or} \\
Q = \left[ \frac{\alpha \sqrt{S}}{n} \left( \frac{\tau}{\rho g S} \right)^{5/3} \right]^{1/1-\beta} \\
Q_c = \left[ \frac{\alpha \sqrt{S}}{n} \left( \frac{\tau_c}{\rho g S} \right)^{5/3} \right]^{1/1-\beta}
\]
Conservation of Energy

- Units: Length
- Head 'loss'
- H, EGL
- Components

\[
z_1 + h_1 + \frac{U_1^2}{2g} = z_2 + h_2 + \frac{U_2^2}{2g} + h_f
\]

or

\[H_1 = H_2 + h_f\]
• **Gradually varied flow**
  Estimate \( h_f \) use energy eqn. to estimate depth at one place given depth at another place (next)

• **Rapidly varied flow**
  (gates, weirs, rapid changes in width or bottom elevation)
  flow often complex, but can be approximated by assuming negligible energy loss (*if flow converging*)
  Over short distances, with small energy loss,

\[
E = h + \frac{U^2}{2g}
\]

\[
z_1 \approx z_2 \quad \text{and} \quad h_f \approx 0, \quad \text{so}

\[
z_1 + h_1 + \frac{U_1^2}{2g} = z_2 + h_2 + \frac{U_2^2}{2g} + h_f
\]

becomes

\[
h_1 + \frac{U_1^2}{2g} = h_2 + \frac{U_2^2}{2g}
\]

or

\[
E_1 = E_2
\]
The Specific Energy $E$

flow energy relative to the bed

$$E = h + \frac{U^2}{2g}$$

$Q = UA$  For a rect. channel

$Q = Ubh$  Define

$q \equiv \frac{Q}{b} = Uh$

so $U = \frac{q}{h}$ and

$$E = h + \frac{q^2}{2gh^2}$$
The extraordinary properties of the Specific Energy $E$

- 2 flow states
- Change in $h$ with step or constriction
- Properties of $\text{Min}(E)$
  \[ F = 1 \]

\[ F = \frac{U}{\sqrt{gh}} = \frac{q}{\sqrt{gh^3}} = 1 \]

To find the minimum $E$, set $\frac{dE}{dh} = 0$.

\[ \frac{d}{dh} \left( h + \frac{q^2}{2gh^2} \right) = 0 \]

\[ 1 - \frac{g^2}{gh^3} = 0 \]

\[ \frac{q^2}{gh^3} = \frac{U^2}{gh} = 1 = F^2 \]

so \( F = 1 \) at minimum $E$.

Note, too, that $q = \sqrt{gh^3}$ where a subscript 'c' is included to indicate that this holds at critical flow.
What happens when flow goes over a smooth step of height $\Delta z$, or through a contraction in width, with negligible head loss ($H$ const)?

**Flow Over Step**

\[
\begin{align*}
  z_1 + \Delta z &= z_2 \quad \text{and} \quad h_f \approx 0, \\
  z_1 + E_1 &= z_2 + E_2 \\
  \text{or} \quad E_1 - \Delta z &= E_2
\end{align*}
\]

**Flow Through Contraction $E_1 = E_2$**

\[
E = h + \frac{q^2}{2gh^2}
\]

\[
F = \frac{U}{\sqrt{gh}} = 1
\]

Diagrams from Roberson & Crowe, Fluid Mechanics, Wiley

Diagrams use $y$ for flow depth and $V$ for velocity
Gradually Varied Flow: Modeling Water Surface Profiles
aka backwater or step-backwater modeling

Switching back depth from $y$ to $h$ and mean velocity from $V$ to $U$ .... Sorry!

Conservation of energy in open channel flow

\[ z_1 + h_1 + \frac{U_1^2}{2g} = z_2 + h_2 + \frac{U_2^2}{2g} + h_f \]

Long distances: $Z_1 > Z_2$ and $h_f > 0$
The rate of head loss is defined as $S_f$

Must estimate head loss, so we need length of stream over which the head loss is occurring

$$\Delta x = (x_2 - x_1)$$

$$z_1 + h_1 + \frac{U_1^2}{2g} = z_2 + h_2 + \frac{U_2^2}{2g} + h_f$$

$$\left( z_2 + h_2 + \frac{U_2^2}{2g} \right) - \left( z_1 + h_1 + \frac{U_1^2}{2g} \right) = -h_f = -S_f \Delta x$$

$$\frac{\left( z_2 + h_2 + \frac{U_2^2}{2g} \right) - \left( z_1 + h_1 + \frac{U_1^2}{2g} \right)}{x_2 - x_1} = \frac{\Delta H}{\Delta x} \equiv -S_f$$

Just restating the definition of energy in open channel flow
Using \[ \frac{z_2 - z_1}{x_2 - x_1} = -S_o \]

becomes

\[
\frac{(z_2 + h_2 + \frac{U_2^2}{2g}) - (z_1 + h_1 + \frac{U_1^2}{2g})}{x_2 - x_1} = \frac{\Delta H}{\Delta x} = -S_f
\]

\[
\frac{(h_2 + \frac{U_2^2}{2g}) - (h_1 + \frac{U_1^2}{2g})}{x_2 - x_1} = S_o - S_f
\]

An ordinary differential equation that tells us the rate \((\Delta h/\Delta x)\) at which \(h\) changes alongstream. This relation only helps if we know the depth at some place to start with. If we do, then the equation can be used to calculate depth at some other location and we are off to the races.
Given $h_1$ at $x_1$, at what $x_2$ will $h_2$ occur?

**Direct Step**

Specify $Q$, $S_o$, and $h_1$.
Continuity gives $U_1$
Pick $h_2$, continuity gives $U_2$
Solve for $x_2$.

$$ \left( h_2 + \frac{U_2^2}{2g} \right) - \left( h_1 + \frac{U_1^2}{2g} \right) = S_o - S_f $$

$$ x_2 = x_1 + \frac{ \left( h_2 + \frac{U_2^2}{2g} \right) - \left( h_1 + \frac{U_1^2}{2g} \right) }{ S_o - S_f } $$

Not very convenient, except for prismatic canals!
Given $h_1$ at $x_1$, what is $h_2$ at $x_2$?

$$\left( h_2 + \frac{U_2^2}{2g} \right) - \left( h_1 + \frac{U_1^2}{2g} \right) = \frac{S_o - S_f}{x_2 - x_1}$$

Specify $Q$, $S_o$, $h_1$ and $\Delta x$

Continuity gives $U_1$

**Guess** $h_2$, continuity gives $U_2$

Iterate.
Defining $S_f$. We used Manning’s eqn to describe flow resistance

For steady, uniform flow, $S_o = S_f$ and we did not have to specify which $S$ was used in Manning’s eq.

We now define that slope as $S_f$ and we use Manning’s eq. to define $S_f$.

$$h_2 = h_1 + \frac{1}{2g} \left( U_1^2 - U_2^2 \right) \left( S_o - S_f \right)(x_2 - x_1)$$

Having chosen a value of $h_2$, we calculate $R_2$, $U_2$, and $S_f$.

Inserting $U_2$ and $S_f$ on the right side, we then calculate a new, improved guess for $h_2$.

This is step-backwater modeling. This is HEC-RAS.
Big advantages of a canned program:

(a) Iterative solution to the governing equation is not always stable

(b) The world is messy (irregular cross-sections, variable roughness, compound channels, bridges, culverts, ....)

Don’t believe a RAS solution unless you can sketch it yourself! How?

(1) identify controls (known depth for a specified discharge) critical depth, normal flow

(2) which way to go (start D/S in subcritical flow and U/S in supercritical flow

(3) Sketch!

In its most basic application, to build a HEC-RAS model you (a) survey cross sections in your study reach and enter these into RAS, (b) assign roughness to each section (or portions of sections), (c) specify a discharge, and (d) step back and let it run.

ROUGHNESS: If you don’t calibrate, you are only guessing.
Types of gradually varied flow profiles

Subcritical Flow – Downstream Control!

Supercritical Flow – Upstream Control!

\[ h_c = \left( \frac{q^2}{g} \right)^{1/3} \]
aka F = 1

\[ h_n = \left( \frac{nq}{\sqrt{S}} \right)^{3/5} \]
aka Manning’s eq.

Types of gradually varied flow profiles:
- **Mild Slope**
  - Backwater
  - M1
  - \( h_n \)
  - \( h_c \)
  - M2
  - \( h_n \)
  - M3
  - \( h_n \)
  - Hydraulics Jump
- **Steep Slope**
  - Backwater
  - S1
  - \( h_c \)
  - \( h_n \)
  - S2
  - \( h_c \)
  - S3
  - \( h_c \)
  - Hydraulics Jump

Subcritical Flow – Downstream Control!

Supercritical Flow – Upstream Control!
### Some Key Points

<table>
<thead>
<tr>
<th>How often does sediment move?</th>
<th>What is the Sediment Balance?</th>
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</thead>
<tbody>
<tr>
<td>Flow <strong>COMPETENCE</strong> of bed material</td>
<td>Sediment Supply <em>versus</em> Transport <strong>CAPACITY</strong></td>
</tr>
<tr>
<td>Threshold Channel, Incipient motion</td>
<td>Alluvial channel problem</td>
</tr>
<tr>
<td>Shields curve + flood frequency</td>
<td>Transport relations give capacity. Supply?</td>
</tr>
</tbody>
</table>

All sediment transport problems have a FLOW PART

- **Flow problem** → hydraulics → boundary stress $\tau_o$
- **Sediment problem** → sediment response to boundary stress $\tau_o$

All relations are approximate. You can quantitatively evaluate the error associated with approximations (e.g. $\tau_o = \rho g RS$)

Channel hydraulics and transport: uncertainty is in the input! can estimate the magnitude and importance of uncertainty need actual measurements to obtain accuracy

The tools are general, their application is local.

**Why do I have to learn RAS?**

- Provides useful basis for estimating flow and stress
- Accounts (coarsely) for nonuniform flow conditions
(1) Finding normal depth

(1a) Finding Normal Depth - Rectangular Channel

\[ V = \frac{R_h^{2/3}}{n} \sqrt{S_o} \quad \text{and} \quad Q = \frac{A^{5/3}}{nP^{2/3}} \sqrt{S_o} \]

where

\[ A = bh, \quad P = b + 2h \quad , \quad R_h = \frac{bh}{b + 2h} \]

Given \( R_h, n, Q \) \( \Rightarrow \) Find \( S_o \) from \( \oplus \)

Given \( R_h, n, S_o \) \( \Rightarrow \) Find \( Q \) from \( \oplus \)

Given \( R_h, Q, S_o \) \( \Rightarrow \) Find \( n \) from \( \oplus \)

Given \( S_o, n, Q \) \( \Rightarrow \) To find \( h \), have to iterate!

Try \( \oplus \) arranged as \( Q = \frac{b}{n} \frac{h^{5/3}}{(1 + 2\frac{h}{b})^{2/3} \sqrt{S_o}} \)

Solve for \( y \) on top:

\[ h_{n+1} = \left( \frac{nQ}{b \sqrt{S_o}} \right)^{3/5} \left( 1 + 2\frac{h_n}{b} \right)^{2/5} \]

where \( n \) and \( n + 1 \) subscripts indicate successive approximations.
(1b) Finding Normal Depth - Trapezoidal Channel

Manning's eqn: 

\[ Q = \frac{\sqrt{S_o}}{n} \left[ \frac{h(B_o + sh)}{B_o + 2h\sqrt{1 + s^2}} \right]^{2/3} \]

where

\[ B = B_o + 2sh \quad A = B_o h + sh^2 = h(B_o + sh) \quad P = B_o + 2h\sqrt{1 + s^2} \]

Given \( h, n, Q \) \( \Rightarrow \) Find \( S_o \) from \( \otimes \)

Given \( h, n, S_o \) \( \Rightarrow \) Find \( Q \) from \( \otimes \)

Given \( h, S_o, Q \) \( \Rightarrow \) Find \( n \) from \( \otimes \)

Given \( S_o, n, Q \) \( \Rightarrow \) To find \( h \), have to iterate!

Try \( \otimes \) arranged as

\[ h_{n+1} = \left( \frac{nQ}{\sqrt{S_o}} \right)^{3/5} \left[ \frac{B_o + 2h_n\sqrt{1 + s^2}}{B_o + sh_n} \right]^{2/5} \]

where \( n \) and \( n + 1 \) subscripts indicate successive approximations
(2) Finding critical depth

Finding Critical Depth \( F = 1 = \frac{V}{\sqrt{gh}} = \frac{Q/A}{\sqrt{g(A/B)}} \) or \( \frac{Q}{\sqrt{g}} = \frac{A^{3/2}}{\sqrt{B}} \)

where \( B \) is top width.

(2a) Rectangular Channel \( \frac{Q}{\sqrt{g}} = \frac{A^{3/2}}{\sqrt{B}} = \frac{(Bh)^{3/2}}{\sqrt{B}} \) or \( \bar{h}_c = \left( \frac{Q}{\sqrt{gB}} \right)^{2/3} \)

(2b) Trapezoidal Channel

\[ B = B_o + 2sh \quad A = B_oh + sh^2 = h(B_o + sh) \]

\[ \frac{A^{3/2}}{\sqrt{B}} = \frac{h^3(B_o + sh)^3}{B_o + 2sh} \]

\[ \frac{Q}{\sqrt{g}} = \left( \frac{y^3(B_o + sh)^3}{B_o + 2sh} \right)^{1/3} \]

Given \( h_c, B_o \) \( \Rightarrow \) Find \( Q = \frac{gh_c^3(B_o + sh_c)^3}{B_o + 2sh_c} \)

Given \( Q, B_o \) \( \Rightarrow \) Find \( y_c = \left( \frac{Q^2}{g} \frac{B_o + 2sh_c}{(B_o + sh_c)^3} \right)^{1/3} \)

Given \( h_c, Q \) \( \Rightarrow \) Find \( B_o = \frac{gh_c^3}{Q^2} \left( B_o + sh_c \right)^3 - 2sh_c \)